Commun Nonlinear Sci Numer Simulat 97 (2021) 105752

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

A new approach for pricing discounted American options

Tsvetelin S. Zaevski

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. Georgi Bonchev Str., Block 8, Sofia 1113, Bulgaria

A R T I C L E I N F O

Article history:
Received 25 September 2020
Revised 20 January 2021
Accepted 1 February 2021
Available online 3 February 2021

JEL classifications:
C41
G12
G13

MSC:
35R35
35Q91
60G44
91G20

Keywords:
Optimal stopping
Discounted American options
Optimal exercise boundary
Free boundary problems
Perpetual options

A B S T R A C T

The purpose of this paper is to present a new numerical approach for finding the early exercise boundary of discounted American options, which payment structure is generalized by adding an additional discount factor. We approximate this boundary by exponent of piecewise linear functions maximizing the option's holder utility. Once we have derived the exercise boundary, the free boundary problem, which describes the American derivative price, turns to a partial differential equation (PDE) with known boundaries. This PDE is an extension of the Kolmogorov's equation and can be converted to the heat equation, for which many numerical solvers are available. We present two different Monte Carlo methods and a finite difference approach to obtain the fair option prices and give some numerical examples. In the perpetual case we derive closed form formulas.

© 2021 Elsevier B.V. All rights reserved.

1. Introduction

The American style derivatives are a specific extension of the European derivatives, where the latter give the holder the right to receive some previously defined amount $N(x)$ if at the maturity date T the underlying asset has value $S_T = x$. In that way the set of moments at which the exercise is permitted is simply $\mathcal{U} = \{T\}$. The generalization which can be made is to enlarge the set $\mathcal{U}$. If the set $\mathcal{U}$ is discrete we have a Bermuda derivative, whereas if the set is $\mathcal{U} = [0, T]$ we have an American derivative. Since the holder can exercise at any moment we have to define the payment structure $N(t, x)$. It represents the amount which the holder will receive if he decides to exercise at moment t given the underlying asset has value $S_t = x$. If the derivative is an American call option the holder has the right to buy the underlying asset at a previously defined price K in an arbitrary moment before the maturity. The American put gives the holder the right to sell. Thus the payment structures are $N(t, x) = (x - K)^+$ and $N(t, x) = (K - x)^+$, respectively. We introduce a time dependence by a new discount factor. This way the payment structures turn to $N(t, x) = e^{-\lambda t}(x - K)^+$ and $N(t, x) = e^{-\lambda t}(K - x)^+$, i.e. the holder has a benefit to exercise earlier. In such a way these options are related to the financial instruments which

E-mail addresses: t_s_zaevski@math.bas.bg, t_s_zaevski@abv.bg

https://doi.org/10.1016/j.cnsns.2021.105752
1007-5704/© 2021 Elsevier B.V. All rights reserved.

lose some value over the time. Also, in Zaevski [52], proposition 2.3, is proven that a model with a continuous dividend payment can be written as a non-dividend model, but with new parameters. Hence, we can assume that the dividend rate is zero without loss of generality. Note also that the mentioned above change of parameters may lead to a negative risk-free rate. The set $[0, T] \times \mathbb{R}^+$ which consists of all admissible values (t, S_t), can be divided into two parts. In the first part the immediate exercise is optimal, whereas in the second one keeping the derivative is preferable. The boundary between them is called early exercise boundary, or for the sake of simplicity only exercise boundary. We prove that, similarly to the undiscounted case (i.e. $\lambda = 0$), the points below the exercise boundary are optimal for the put options, whereas these above the boundary are optimal for the call options. The undiscounted case of an American call is specific – we give a brief proof of the well known fact that early exercising is never optimal. This means that the early exercise boundary is the infinity. When the discount factor is positive early exercising is possible. The both facts are confirmed by the presented numerical results.

Since the holder has the right to choose the exercise moment it is natural to assume that he will follow a strategy which maximizes his profit. Thus we reach to a problem for optimal stopping – see Lamberton and Lapeyre [30] or Wong [45]. These problems are solutions of the so called free boundary or obstacle problems – see Jacka [20], Kim [26], Pascucci [41], or Magirou et al. [38] – for which we know the differential equation and have to find its solution as well as the region in which it holds. Deriving a closed form solution is hard and more often impossible. For this many authors suggest different numerical solutions. Cox et al. [14] suggest a very useful one based on the binomial trees. Other interesting numerical methods are proposed by Barone-Adesi and Whaley [4], Bjerksund and Stensland [6], Geske and Johnson [16], Ho et al. [18], Johnson [22], Ju [23], and Longstaff and Schwartz [35]. We mention also the works of Ju [23] and Huang et al. [19]. They use a recursive method based on the classical works of Jacka [20], Kim [26], and Carr et al. [9].

In addition, many authors suggest new approaches for pricing American derivatives in the recent years. Rad et al. [42] use a radial basis point interpolation method. Yoon [47] examines perpetual options with a stochastic elasticity of the variance. Yu and Xie [49] present an entropy based model with incorporated risk-neutral moments. Backward stochastic differential equations are used in Klimsiak et al. [27]. An approach based on an integral equation is applied in Le and Dang [31] to price American type Parisian down-and-out call options. The Laplace-Carson transform is used for pricing different American style options in Park and Jeon [40] and Kang et al. [24]. Three methods for pricing American options in the presence of stochastic volatility are presented in Balajewicz and Toivanen [3], Gong and Zhuang [17], and Kozpınar et al. [28] – there are introduced in addition jumps in the first two papers. Other models which exhibit a jump behavior are presented in Company et al. [13] and Yang [46]. Chen et al. [12] use stable processes in the Black-Scholes framework for American options. A consumption based approach is provided in Alghalith [1]. In Zhao and Yang [54] are presented semi-smooth Newton methods for pricing American options. Systems of partial integro-differential equations are used in Yousuf et al. [48]. The Brennan-Schwartz algorithm is applied to American option pricing in Madi et al. [37]. Zaevski [51] derived a new form of the early exercise premium. Some Fourier series methods are used in Chan [10] and Chan [11]. Armerin [2] introduces American style options which life begins after some random event (stopping time). An asymptotic expansion approach is presented in Li and Ye [33]. Gao et al. [15] examine American better-of options on two assets. The pricing problem for American power put options written on a non-dividend underlying asset is examined in Lee [32]. Models with a two state regime switching volatility can be found in Lu and Putri [36] and Jeon et al. [21]. An integral equation approach is used for pricing American barrier options in Lin and He [34].

There are two main questions which interest the holder of an American derivative – what is the fair price of the derivative and is the immediate exercise optimal or not. We first solve the second problem approximating the early exercise boundary. We use exponent of piecewise linear functions imposing continuity at the nodes. We derive these functions maximizing the expected payoffs of specific derivatives. These derivatives have the same payment structure as the corresponding option, but the exercise moment is set to be the first hitting time to the previously defined boundary. Once we derive the early exercise boundary we can view the free boundary problem as a boundary value problem for a PDE with known boundary constrains. This PDE can be converted to a heat equation for which there exist many numerical solvers. We give two Monte Carlo methods which lead to the prices of the discounted American options. We compare some prices of undiscounted American put options derived by our approach with the corresponding prices derived by other existing methods. In such a way we validate our results. In addition we present an approach based on a finite differences approximation of the PDE. Applying it for some call style options we ascertain that it works very fast and accurate.

We also examine the perpetual case in which there is not maturity date, i.e. it is infinity – $T = \infty$ and $\mathcal{U} = [0, \infty)$. Since the asset price is presented by a Markov process, the future behavior of the asset is affected only from its current value. Also, since we have not a maturity horizon we are not threatened by a forced exchange at the maturity. This means that we are able to wait the asset price to reach the optimal value which is independent of the time. Hence, the exercise boundary is flat. Using our approach for maximizing the future payoffs we derive closed form formulas for the early exercise boundaries. After that we derive closed form formulas for the discounted American option prices too.

The paper is organized as follows. In Section 2 we provide the base we shall use later and in Section 3 we define the exercise regions and prove some important propositions. In Sections 4 and 5 are given the algorithms for deriving the early exercise boundary and the price of American put and call options, respectively. Some numerical examples are presented too. In Section 6 we obtain the closed form formulas for the perpetual discounted American options.

2. Preliminaries

Although our examination is based on the geometric Brownian motion, some results are true under more general assumptions. For this we begin without Gaussian restrictions and later we shall impose them. Let the asset price S_t be driven by a Feller–Markov process under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ which satisfies the usual conditions.[1] The measure Q is assumed to be risk neutral. We shall denote by $E^{t,x}$ the expectation assuming that the underlying asset value at the moment t is x. If $t = 0$ we shall use the notation E^x. Suppose for simplicity that the risk-free rate of return is a constant r. Let the function $N(t, x)$ defines the amount which the holder will receive if he decides to exercise at a moment t and the current asset price is $S_t = x$. Let $\mathcal{T}_{[t,T]}$ be the set of all stopping times τ such that $t \le \tau \le T$. The set $\{(t, x) : 0 \le t \le T, x \ge 0\}$ can be divided into two parts – continuation and exercise regions. In the first one keeping the derivative is preferable, whereas in the second one the immediate exercise is optimal. We shall denote by Υ the exercise region, by $\overline{\Upsilon}$ the continuation region, and by $c(t)$ the exercise boundary. Let τ be the exercise moment and Λ_t and Φ_t be the indicator processes $\Lambda_t = I_{\{\tau \le t\}}$ and $\Phi_t = I_{\{t<\tau\}} \equiv 1 - \Lambda_t$. Let $A(t, x)$ be the function which determines the price of the American derivative, namely $A(t, S_t)$. As we mentioned above, it is possible the risk-free rate r to be negative. On the contrary, we assume that the discount rate is positive, $\lambda > 0$, as well as the total discount rate is positive, $\lambda + r > 0$. The last requirement guarantees that the financial derivative has a decreasing behavior after the total discounting (with $\lambda + r$ rate).

We shall impose the following natural condition.

Condition 2.1. Under the same other conditions, the future asset value is larger when its initial value is larger. Something more, if the initial asset value tends to zero/infinity, then the asset value tends to zero/infinity at every moment. The convergence is pathwise.

The following proposition presents the asset's behavior w.r.t. its initial value.

Proposition 2.1. *Let Condition 2.1 be satisfied and $\zeta \in \mathcal{T}_{[t,T]}$ be a stopping time. Then we have*

1. *If $f(\cdot,\cdot)$ is a non-decreasing w.r.t. the second variable function and $0 < y < x$, then $E^{t,y}\left[e^{-r\zeta} f\left(\zeta, S_\zeta\right)\right] < E^{t,x}\left[e^{-r\zeta} f\left(\zeta, S_\zeta\right)\right]$.*
2. *If $f(\cdot,\cdot)$ is a continuous w.r.t. the second variable function and $x \to 0$, then $E^{t,x}\left[e^{-r\zeta} f\left(\zeta, S_\zeta\right)\right] \to E^{t,0^+}\left[e^{-r\zeta} f(\zeta, 0)\right]$.*
3. *If $f(\cdot,\cdot)$ is a continuous w.r.t. the second variable function and $x \to \infty$, then $E^{t,x}\left[e^{-r\zeta} f\left(\zeta, S_\zeta\right)\right] \to E^{t,\infty}\left[e^{-r\zeta} f(\zeta, \infty)\right]$.*

Something more, if $f(t, x) = x$, then Condition 2.1 is not necessary.

Proof. The proof is obvious. We shall examine only the case $f(x) = x$ without Condition 2.1. Since the process $e^{-rt}S_t$ is a martingale,

$$E^{t,y}\left[e^{-r\zeta} S_\zeta\right] = e^{-rt}y < e^{-rt}x = E^{t,x}\left[e^{-r\zeta} S_\zeta\right].$$

The proofs of the second and third statements are analogous. □

3. Exercise region

The exercise and continuation regions are defined as

1. The point $(t, x) \in \Upsilon$ if for all stopping times $\zeta \in \mathcal{T}_{[t,T]}$,

$$N(t, x) \ge E^{t,x}\left[e^{-r(\zeta-t)} N\left(\zeta, S_\zeta\right)\right]. \tag{3.1}$$

2. The point $(t, x) \in \overline{\Upsilon}$ if there exists a stopping time $\zeta \in \mathcal{T}_{[t,T]}$, such that

$$N(t, x) < E^{t,x}\left[e^{-r(\zeta-t)} N\left(\zeta, S_\zeta\right)\right]. \tag{3.2}$$

Now we turn to the defined above discounted American options. The corresponding payments for the call and put options are, respectively,

$$N(t, x) = e^{-\lambda t}(x - K)^+ \tag{3.3}$$

$$N(t, x) = e^{-\lambda t}(K - x)^+. \tag{3.4}$$

We shall prove some propositions for the form of the exercise region.

Proposition 3.1. *In the case of undiscounted American call options, $\lambda = 0$, we have that $\Upsilon = \emptyset$, i.e. early exercising is never optimal.*

[1] The filtration is right continuous and complete.

Proof. Since the total discount rate is positive, $r+\lambda>0$, and $\lambda=0$, we have that $r>0$. Suppose that $(t,x)\in\Upsilon$ and the stopping time $\zeta\in\mathcal{T}_{[t,T]}$ is arbitrary. Obviously $x=S_t>K$ and therefore, since $e^{-rt}S_t$ is a martingale, we have

$$\begin{aligned}E^{t,x}\left[e^{-r\zeta}N\left(\zeta,S_\zeta\right)\right]&=E^{t,x}\left[e^{-r\zeta}\left(S_\zeta-K\right)^+\right]\\&\le e^{-rt}(x-K)\\&=E^{t,x}\left[e^{-r\zeta}S_\zeta\right]-Ke^{-rt}\\&<E^{t,x}\left[e^{-r\zeta}\left(S_\zeta-K\right)\right]\\&\le E^{t,x}\left[e^{-r\zeta}\left(S_\zeta-K\right)^+\right].\end{aligned}$$

The contradiction finishes the proof. □

Differently from the undiscounted case, the exercise region for an American call is not empty in the presence of a discount factor. This can be seen from the following proposition.

Proposition 3.2. *If $\lambda>0$, the early exercise region Υ is not empty. Something more, if $(t,x)\in\Upsilon$ and $y>x$, then (t,y) is also in Υ.*

Proof. The third statement of Proposition 2.1 shows that for a large enough x the inequality

$$\begin{aligned}&E^{t,x}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta-K\right)^+\right]-e^{-(r+\lambda)t}(x-K)\\&=E^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(-K-S_\zeta\left(e^{\lambda(\zeta-t)}-1\right),-e^{\lambda(\zeta-t)}S_\zeta\right)\right]+e^{-(r+\lambda)t}K\\&=E^{t,x}\left[e^{-(r+\lambda)\zeta}\max\begin{pmatrix}K\left(e^{(r+\lambda)(\zeta-t)}-1\right)-S_\zeta\left(e^{\lambda(\zeta-t)}-1\right),\\-e^{\lambda(\zeta-t)}S_\zeta+e^{(r+\lambda)(\zeta-t)}K\end{pmatrix}\right]\le 0\end{aligned}$$

is true for every $\zeta\in\mathcal{T}_{[t,T]}$ and therefore $(t,x)\in\Upsilon$. Hence the set Υ is not empty. Now suppose that $y>x$ and $(t,x)\in\Upsilon$. First, note that $y>x>K$. Let $\zeta\in\mathcal{T}_{[t,T]}$ be arbitrary. Using the first statement of Proposition 2.1 we derive

$$\begin{aligned}&E^{t,y}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta-K\right)^+\right]-e^{-(r+\lambda)t}(y-K)\\&=E^{t,y}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta-K\right)^+\right]-e^{-\lambda t}E^{t,y}\left[e^{-r\zeta}S_\zeta\right]+e^{-(r+\lambda)t}K\\&=E^{t,y}\left[e^{-(r+\lambda)\zeta}\max\left(-K-S_\zeta\left(e^{\lambda(\zeta-t)}-1\right),-e^{\lambda(\zeta-t)}S_\zeta\right)\right]+e^{-(r+\lambda)t}K\\&\le E^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(-K-S_\zeta\left(e^{\lambda(\zeta-t)}-1\right),-e^{\lambda(\zeta-t)}S_\zeta\right)\right]+e^{-(r+\lambda)t}K\\&=E^{t,x}\left[e^{-(r+\lambda)\zeta}\left(S_\zeta-K\right)^+\right]-e^{-(r+\lambda)t}(x-K)\le 0.\end{aligned}$$

Therefore $(t,y)\in\Upsilon$ too. □

When the option is put style, the following proposition determines its exercise region.

Proposition 3.3. *The set Υ is not empty. Let $(t,x)\in\Upsilon$ and $y<x$. Then (t,y) is also in Υ.*

Proof. We have for an arbitrary $\zeta\in\mathcal{T}_{[t,T]}$

$$\begin{aligned}&E^{t,x}\left[e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+\right]-e^{-(r+\lambda)t}(K-x)\\&=E^{t,x}\left[e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+\right]+e^{-\lambda t}E^{t,x}\left[e^{-r\zeta}S_\zeta\right]-e^{-(r+\lambda)t}K\\&=E^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(K+\left(e^{\lambda(\zeta-t)}-1\right)S_\zeta,e^{\lambda(\zeta-t)}S_\zeta\right)\right]-e^{-(r+\lambda)t}K\\&=E^{t,x}\left[e^{-(r+\lambda)\zeta}\max\begin{pmatrix}-\left(e^{(r+\lambda)(\zeta-t)}-1\right)K+\left(e^{\lambda(\zeta-t)}-1\right)S_\zeta,\\x-e^{(r+\lambda)(\zeta-t)}K+e^{\lambda(\zeta-t)}S_\zeta\end{pmatrix}\right].\end{aligned}\tag{3.5}$$

We use the second statement of Proposition 2.1 to conclude that for a small enough x expression (3.5) is not positive and therefore the set Υ is not empty.

Suppose now that $(t,x) \in \Upsilon$ and $y < x < K$. Let $\zeta \in \mathcal{T}_{[t,T]}$ be arbitrary. We shall use again the first statement of Proposition 2.1. We have

$$\begin{aligned}
&E^{t,y}\left[e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+\right] - e^{-(r+\lambda)t}(K-y) \\
&= E^{t,y}\left[e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+\right] + e^{-\lambda t}E^{t,y}\left[e^{-r\zeta}S_\zeta\right] - e^{-(r+\lambda)t}K \\
&= E^{t,y}\left[e^{-(r+\lambda)\zeta}\max\left(K+S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), e^{\lambda(\zeta-t)}S_\zeta\right)\right] - e^{-(r+\lambda)t}K \\
&\le E^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(K+S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), e^{\lambda(\zeta-t)}S_\zeta\right)\right] - e^{-(r+\lambda)t}K \\
&= E^{t,x}\left[e^{-(r+\lambda)\zeta}\left(K-S_\zeta\right)^+\right] - e^{-(r+\lambda)t}(K-x) \le 0.
\end{aligned}$$

Therefore $(t,y) \in \Upsilon$. □

Remark 3.1. Propositions 3.1 –3.3 can be proved in the perpetual case by letting T to tend to infinity.

From now on we assume that the asset price is a geometric Brownian motion under the risk-neutral measure

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t\right). \tag{3.6}$$

The next step is to determine the value of the exercise boundary at the maturity. We shall use a similar approach to one presented in Kwok [29], page 257.

Proposition 3.4. *The value of the exercise boundary at the maturity for a call option is given by*

$$c(T) = \max\left(\frac{r+\lambda}{\lambda}, 1\right)K. \tag{3.7}$$

Proof. We have to derive the lower value of x which belongs to the exercise region near the maturity. Note that it is not below the strike. Suppose that $(t,x) \in \Upsilon$ and therefore the option value is given by the payment function

$$A(t,x) = N(t,x) = e^{-\lambda t}(x-K). \tag{3.8}$$

Also, as a consequence of the definition of the exercise region and inequality (3.1) we derive the corresponding variational inequality

$$N_t(t,x) + \mathcal{A}N(t,x) - rN(t,x) \le 0. \tag{3.9}$$

We denote above by $\mathcal{A}$ the infinitesimal generator of log-normal process (3.6)

$$\mathcal{A}f(x) = rxf'(x) + \frac{1}{2}\sigma^2 x^2 f''(x). \tag{3.10}$$

Inequality (3.9) is equivalent to

$$\overline{K} \equiv \frac{r+\lambda}{\lambda}K \le x. \tag{3.11}$$

Suppose that the risk free rate is positive, $r > 0$, or equivalently $\overline{K} > K$. We shall prove that the exercise boundary at the maturity is just $\overline{K}$. Suppose that this is not true, i.e. the exercise boundary at the maturity is larger than $\overline{K}$. This means that there exists $x > \overline{K}$ which belongs to the continuation region near the strike. We shall use that $A(t,x) > N(t,x)$ in the continuation region and that $A(t,x)$ solves the Black–Scholes equation. Taking in attention that for $x > \overline{K} > K$ the function $N(T,x)$ is differentiable, we derive

$$\begin{aligned}
0 &< \lim_{t\to T} \tfrac{A(t,x)-N(t,x)}{T-t} \\
&= -\lim_{t\to T} \tfrac{A(T,x)-A(t,x)}{T-t} + \lim_{t\to T} \tfrac{N(T,x)-N(t,x)}{T-t} \\
&= \mathcal{A}A(T,x) - rA(T,x) + N_t(T,x) \\
&= rxe^{-\lambda t} - re^{-\lambda t}(x-K) - \lambda e^{-\lambda t}(x-K) \\
&= e^{-\lambda t}[(r+\lambda)K - \lambda x] < 0.
\end{aligned} \tag{3.12}$$

The contradiction shows that the exercise boundary can not be higher than the critical value $\overline{K}$.

If the risk-free rate is negative, $r < 0$ (equivalent to $\overline{K} < K$) the same reasons as above shows that the exercise boundary can not be larger than the strike. This finishes the proof. □

Remark 3.2. If the discount rate is zero (and therefore $r > 0$), exercise boundary (3.7) is infinity. This corresponds to the fact that early exercising is never optimal (Proposition 3.1).

We can prove the analogous proposition for the put options.

Proposition 3.5. *The value of the exercise boundary at the maturity for a put option is given by*

$$c(T) = \min\left(\frac{r+\lambda}{\lambda}, 1\right)K. \tag{3.13}$$

Proof. The difference with Proposition 3.4 is that we have to derive the higher value of x, for which variational inequality (3.9) holds. The payment function of the put option is

$$N(t,x) = e^{-\lambda t}(K - x). \tag{3.14}$$

Therefore inequality (3.11) turns to

$$\frac{r+\lambda}{\lambda}K > x. \tag{3.15}$$

We finish the proof analogously to the proof of Proposition 3.4. □

Remark 3.3. Note that if the risk free rate is positive, then the exercise boundary value at the maturity is the strike.

4. The algorithm for pricing discounted American put options

Note that Proposition 3.3 means that the form of the exercise region is $\Upsilon = \{(t,x) : t \in [0,T], x \in (0, c(t)]\}$, where $c(t)$ is the exercise boundary. As we mentioned above, we shall approximate the exercise boundary by exponent of piecewise linear functions. Let the time interval $[0,T]$ be divided into n sub-intervals $0 \equiv t_0 < t_1 < \ldots < t_n \equiv T$. Suppose that the holder strategy τ is to exercise when the asset reaches the level $\exp(a_i t + b_i)$ if this happens in the interval $[t_{i-1}, t_i)$, $i = 1, 2, \ldots, n$. We also state continuity $\exp(a_i t_i + b_i) = \exp(a_{i+1} t_i + b_{i+1}) \equiv C_i$. Assume that the underlying asset starts from the value x, i.e. $S_0 = x$. Therefore the exercise happens when the Brownian motion touches the level

$$\frac{1}{\sigma}\left(\left(a_i - r + \frac{\sigma^2}{2}\right)t + b_i - \log(x)\right) = A_i t_i + B_i \tag{4.1}$$

for

$$\begin{aligned} A_i &= \frac{1}{\sigma}\left(a_i - r + \frac{\sigma^2}{2}\right) \\ B_i &= \frac{b_i - \log(x)}{\sigma}. \end{aligned} \tag{4.2}$$

Let τ be just this moment. Let us define a derivative which pays amount of $\exp(-\lambda(\tau \wedge T))(K - S_{\tau\wedge T})^+$ at the moment $\tau \wedge T$. We denote its price by

$$\begin{aligned} A(x; \{t_0, \ldots, t_n\}; \{C_0, \ldots, C_n\}) &= E^x\left[e^{-(r+\lambda)(\tau\wedge T)}(K - S_{\tau\wedge T})^+\right] \\ &= E^x\left[e^{-(r+\lambda)\tau}(K - S_\tau)\Lambda_T\right] + E^x\left[e^{-(r+\lambda)T}(K - S_T)^+\Phi_T\right] \\ &= KE\left[e^{-\alpha_1\tau}\Lambda_T\right] - x\sum_{m=1}^{n} e^{\sigma B_m} E\left[e^{-\alpha_{2,m}\tau} I_{t_{m-1}<\tau\le t_m}\right] \\ &\quad + Ke^{-\alpha_1 T}P(B_T < k, \Phi_T = 1) - xe^{-\alpha_3 T}E\left[e^{\sigma B_T} I_{B_T<k, \Phi_T=1}\right] \end{aligned} \tag{4.3}$$

for

$$\begin{aligned} \alpha_1 &= r + \lambda \\ \alpha_{2,m} &= (r+\lambda) - \left(r - \frac{\sigma^2}{2}\right) - A_m\sigma = \frac{\sigma^2}{2} - A_m\sigma + \lambda \\ \alpha_3 &= \lambda + \frac{\sigma^2}{2} \\ k &= \tfrac{1}{\sigma}\log\left(\tfrac{K}{x}\right) - \left(\tfrac{r}{\sigma} - \tfrac{\sigma}{2}\right)T. \end{aligned} \tag{4.4}$$

Our algorithm is based on Eqs. (4.3) and (4.4) and Propositions A.3 and A.5 and is as follows

1. The value of the exercise boundary at the maturity, C_n, is given by Eq. (3.13).
2. We shall find the value of the exercise boundary at the previous point t_{n-1}. Let us define $C(x)$ for a fixed $x \le K$ as

$$C(x) = \arg\max\{C : A(x; \{0, t_n - t_{n-1}\}; \{C, C_n\})\}. \tag{4.5}$$

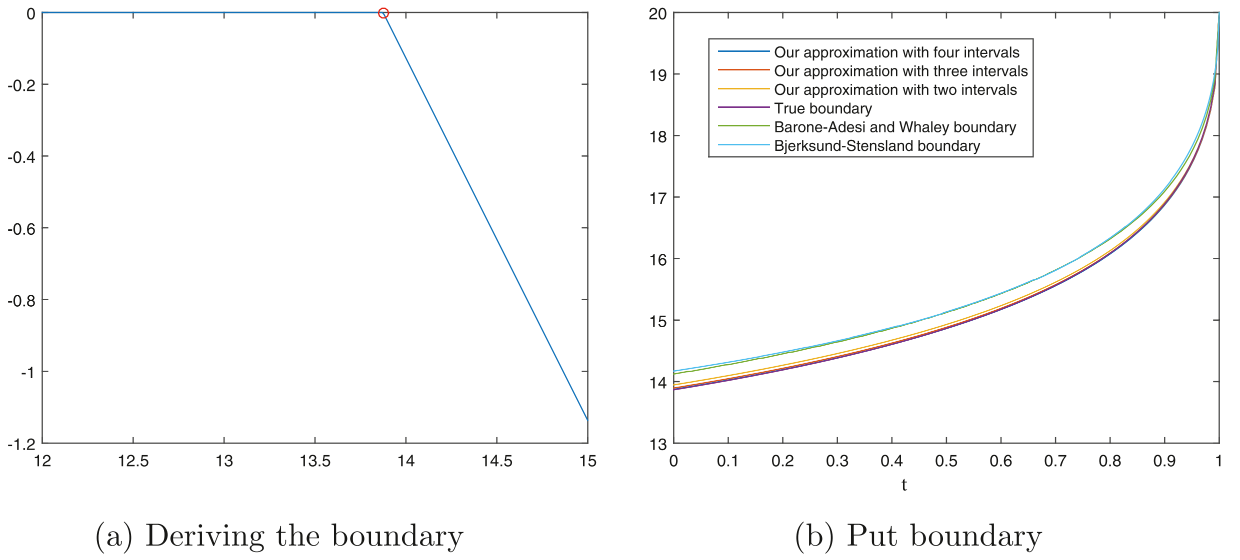


Fig. 1. Early exercise boundary of a put option. The parameters are $r = 0.05$, $\lambda = 0$, $\sigma = 0.3$, $K = 20$, and $T = 1$.

Proposition 3.3 gives that the exercise region at time t_{n-1}, Υ_{n-1}, has the form $[0, c_{n-1}]$. Therefore

$$C_{n-1} = \max\{x : C(x) = x\} \tag{4.6}$$

or equivalently

$$C_{n-1} = \max\{x : A(x; \{0, t_n - t_{n-1}\}; \{C(x), C_n\}) = K - x\}. \tag{4.7}$$

Functions $A(x; \{0, t_n - t_{n-1}\}; \{C, C_n\})$ are calculated by the use of Eqs. (4.3) and (4.4), Propositions A.2 and A.4, and Corollary A.1. Note that if $r > 0$, the last two terms of formula (4.3) vanish.

3. Suppose that we have found values C_m, C_{m+1}, ..., C_n for some $m < n$. We proceed in the same way as above to find the value of C_{m-1}. Let us fix some $x \le K$ and denote by $C(x)$

$$C(x) = \arg\max\{C : A(x; \{0, t_m - t_{m-1}, \dots, t_n - t_{m-1}\}; \{C, C_m, \dots, C_n\})\}. \tag{4.8}$$

Analogously to Eqs. (4.6) or (4.7) we can find C_{m-1} as any of both formulas

$$\begin{aligned} C_{m-1} &= \max\{x : C(x) = x\} \\ C_{m-1} &= \max\{x : A(x; \{0, t_m - t_{m-1}, \dots, t_n - t_{m-1}\}; \{C(x), C_m, \dots, C_n\}) = K - x\}. \end{aligned} \tag{4.9}$$

Functions $A(x; \{0, t_m - t_{m-1}, \dots, t_n - t_{m-1}\}; \{C, C_m, \dots, C_n\})$ are again calculated using Eqs. (4.3) and (4.4) and Propositions A.3 and A.5.

A typical behavior of the function $C(x) - x$ is presented at Fig. 1a. The value of the early exercise boundary is marked by a red point.

Once we derive the exercise boundary, the free boundary problem for a discounted American put turns to the boundary value problem

$$\begin{aligned} &*A_t(t,x) + rxA_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 A_{xx}(t,x) - rA(t,x) = 0 \\ &A(t, c(t)) = \exp(-\lambda t)(K - c(t)),\ t \in [0, T] \\ &A(T, x) = \exp(-\lambda T)(K - x)^+,\ x > c(T). \end{aligned} \tag{4.10}$$

The equation is satisfied in the strip $(t, x) \in \{(0, T) \times (c(t), \infty)\}$ and the boundary constrains are imposed on the lower and the right boundaries. This equation can be inverted to a heat equation for which there exists many numerical solvers. On the other hand it is an extension of the Kolmogorov's equation and its solution is given by

$$\begin{aligned} A(t,x) = &e^{-\lambda t} E^{t,x}\Big[e^{-(\lambda + r)(\tau - t)}(K - c(\tau))\Lambda_T\Big] \\ &+ e^{-\lambda t} E^{t,x}\Big[e^{-(\lambda + r)(T - t)}(K - S_T)^+ \Phi_T\Big]. \end{aligned} \tag{4.11}$$

Since the process is Markov it is enough to examine the case $t = 0$. We shall give two Monte Carlo methods for numerical deriving of the expectations in Eq. (4.11).

4.1. First Monte Carlo method for pricing discounted American put options

We simulate the Brownian motion paths dividing the time interval into N sub-intervals. We generate N normally distributed with zero expectation and standard deviation $\sqrt{T/N}$ random numbers. A Brownian path $\left\{B_{t_i}\right\},\ i=1,\dots,N$ is represented by the cumulative sum. We repeat this H times. For every sample path we calculate the term

$$p_i = \exp\left(-(r+\lambda)\tau_i\right)(K - c(\tau_i)) \tag{4.12}$$

if the sample path falls below the boundary before the maturity, and set it to be

$$p_i = \exp\left(-(r+\lambda)T\right)\left(K - S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T\right)\right)^+ \tag{4.13}$$

otherwise. We calculate price (4.11) as $\left(\sum_{i=1}^{H} p_i\right)/H$.

4.2. Second Monte Carlo method

The second Monte Carlo method is based on the approach presented in Wang and Pötzelberger [44], Section 5. We use it to calculate the integrals in Eqs. (A.3), (A.6), and (A.7). In their original work the hitting time is assumed to be above. To follow the same scheme we have to use the corresponding negative values $\bar{c}_m = -c_m$. This is possible since the Brownian motion is symmetric. To calculate the option price we use formula (4.3). To derive the first two terms we use Proposition A.3. The m'th Laplace transform is obtained by the following steps

1. We generate $m-1$ normal random numbers with zero expectation and standard deviation one. They form the vector u.
2. Let D be the $(m-1)\times(m-1)$ diagonal matrix composed by values $\sqrt{T/N}$ and M be a $(m-1)\times(m-1)$ lower triangle matrix with values ones. Define the vector x as $x = MDu$.
3. We calculate the values of the function

   $$w_j = e^{-\alpha t_{m-1}} L(t_m - t_{m-1}, \alpha; a_m, b_m - x_{m-1})$$

   which appears in the integral (A.3).
4. We derive the values of the function

   $$v_j = v(x_1,\dots,x_{m-1}) = \prod_{i=1}^{m-1} I_{x_i<\bar{c}_i}\left(1 - \exp\left(-\frac{2(\bar{c}_{i-1} - x_{i-1})(\bar{c}_i - x_i)}{\frac{T}{N}}\right)\right).$$

5. We calculate the term $p_j = w_j v_j$.
6. We calculate the truncated Laplace transform repeating the above procedure H times and taking average $\left(\sum_{i=1}^{H} p_i\right)/H$.

The last two terms in Eq. (4.3) are obtained in the same way, taking $m=n$ and changing the term $e^{-\alpha t_{m-1}}L(\cdot)$ in w (step 3) by $e^{-\alpha x_{m-1}}U(\cdot)$ and $e^{-\alpha x_{m-1}}V(\cdot)$, respectively. Note that the second product in integral (A.3) is incorporated when we generate normal random numbers and calculate the corresponding expectation.

4.3. Numerical results

We shall examine first the undiscounted case which allows us to compare the results obtained by our approach with the results derived by several existing methods. After that we shall give some examples with positive values of the discount rate λ. We examine a division of the time interval to 128 sub-intervals. For each node we use our algorithm with three, four, and five steps. In such a way, comparing the computational times and price deviations, we can obtain the optimal algorithm. The strike price is assumed to be $K = 20\$$, the risk-free rate is $r = 0.05$,[2] the volatility is $\sigma = 0.3$, and the time to maturity is $T = 1$. At Fig. 1b we present the early exercise boundary for a non-discounted American put calculated by several methodologies. The benchmark is obtained using the Cox et al. [14] tree method with $N = 10,000$ steps. We choose just the binomial tree method for a benchmark because our method is numerical for deriving the exercise boundary as well as for option pricing. In such a way we can evaluate the accuracy of the method in both aspects. We present also the boundaries obtained by the approaches of Barone-Adesi and Whaley [4] and Bjerksund and Stensland [6]. The corresponding boundary values are calculated as the largest value of x, for which the option price is equal to $K - x$. It can be seen that our boundary obtained by the five steps algorithm is very close to the binomial tree boundary. Also, the boundaries obtained by three and four steps algorithms are also admissible and they are much closer to the binomial tree one in comparison with the boundaries derived by the methods of Barone-Adesi and Whaley [4] and Bjerksund and Stensland [6].

Some option prices are presented in Table 1 – the first reported values are derived by the first Monte Carlo method; right to them we place the prices obtained using the second one.[3] We use values $N = 2000$ and $H = 200,000$. The time

[2] Note that Remark 3.3 says that this risk free value leads to $c(T) = K$ and thus the last term in formula (4.11) vanishes.

[3] Other numerical results with different values of the parameters are presented in Zaevski [50].

Table 1
American put option prices. The parameters are $r = 0.05$, $\lambda = 0$, $\sigma = 0.3$, $K = 20$, and $T = 1$. Our approach is presented by three, four, and five steps for deriving the exercise boundary. Versions with 16, 32, 64, and 128 grid nodes are provided. The first and the second values are the prices obtained by both of Monte Carlo methods. We present also the binomial tree prices, as well as the prices obtained by the methods of Bjerksund and Stensland [6], Longstaff and Schwartz [35], and Barone-Adesi and Whaley [4].

	Binomial tree	Our model – 16 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0774/5.0775	5.0776/5.0783	5.0776/5.0769	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5520/3.5516	3.5521/3.5514	3.5522/3.5522	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9707/1.9691	1.9708/1.9693	1.9701/1.9695	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2910/1.2888	1.2911/1.2891	1.2909/1.2893	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6613/0.6600	0.6614/0.6606	0.6613/0.6601	0.6563	0.6803	0.6758
	Binomial tree	Our model – 32 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0779/5.0778	5.0779/5.0778	5.0778/5.0795	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5534/3.5530	3.5536/3.5528	3.5536/3.5533	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9721/1.9713	1.9725/1.9717	1.9725/1.9706	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2929/1.2922	1.2931/1.2927	1.2929/1.2923	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6629/0.6625	0.6630/0.6627	0.6630/0.6627	0.6563	0.6803	0.6758
	Binomial tree	Our model – 64 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0778/5.0782	5.0783/5.0782	5.0783/5.0789	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5541/3.5533	3.5540/3.5546	3.5540/3.5540	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9733/1.9724	1.9735/1.9729	1.9736/1.9740	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2938/1.2937	1.2939/1.2933	1.2940/1.2940	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6637/0.6630	0.6640/0.6635	0.6637/0.6639	0.6563	0.6803	0.6758
	Binomial tree	Our model – 128 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0782/5.0768	5.0785/5.0778	5.0786/5.0772	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5541/3.5549	3.5545/3.5544	3.5545/3.5547	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9739/1.9729	1.9737/1.9748	1.9737/1.9731	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2943/1.2945	1.2941/1.2929	1.2941/1.2940	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6640/0.6638	0.6640/0.6639	0.6640/ 0.6642	0.6563	0.6803	0.6758

interval is divided to 16, 32, 64, or 128 sub-intervals. We also take the option to be at-the-money, i.e. $S_0 = K = 20$. It is important to note that the second method is significantly faster as well as converges faster. To minimize the pricing error, we calculate the prices 100 times and averaging. In addition to the results obtained by the methods of Barone-Adesi and Whaley [4] and Bjerksund and Stensland [6], we add the prices derived by the numerical approach of Longstaff and Schwartz [35]. The corresponding prices are calculated again 100 times and averaged. It can be seen that our method based on the 128 sub-intervals boundary produces values very closed to the binomial tree prices. The difference between the results derived by three, four, and five steps algorithms is insignificant. Something more, if we use one and the same Monte Carlo simulation for the algorithm with different steps, the produced results are practically identical. The 16, 32, and 64 time interval divisions produce very good results too. Also, note that they are significantly better than the results obtained by methods of Barone-Adesi and Whaley [4], Bjerksund and Stensland [6], and Longstaff and Schwartz [35].

All calculations are performed by the use of MATLAB with Intel i7-10510U (4.9 G)/ 16 GB RAM/ 1000 GB SSD. All computational times are reported in Table 5. We can see that they increase significantly for the algorithms with more steps. On the other hand, this do not lead to distinctly more accurate prices. Hence, we can use the algorithm with three steps and 16 grid nodes.

The behavior of the early exercise boundary for the discounted American puts is presented in Figs. 2a and 3 a. In the first one the time varies between zero and one, whereas for the second one $t \in [1, 20]$. The values of the discount factor λ are between zero and one. At Fig. 3a we plotted the boundary values for the corresponding perpetual options by red points. They are calculated using the closed form formula derived in Section 6, Eq. (6.12). It can be seen that for large enough t the boundary surface tends to the perpetual values. We also present the behavior of the option prices w.r.t. the discount factor and the time at Fig. 4a. These prices are calculated using the second Monte Carlo method. We marked by red points the prices of the corresponding perpetual puts – we use the formula given in Theorem 6.2. It can be seen that for large t the option prices tend to the perpetual ones.

In Table 2 we present some prices for discounted put options. We assume that the risk free rate is negative, $r = -0.03$. This leads to a value of the exercise boundary at the maturity below the strike – see formula (3.13). We vary the time to maturity among $T \in \{0.5, 1, 2, 3\}$, the penalty among $\lambda \in \{0.031, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1\}$. These values

Table 2
American put option prices – discounted case. The used parameters are $r = -0.03,\ \sigma = 0.3,\ K = 20,\ \lambda \in \{0.031, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1\}$, and $T \in \{0.5, 1, 2, 3\}$. The time grid has 16 nodes; the algorithm is with three steps. The second Monte Carlo method is used.

	$\lambda = 0.031$	$\lambda = 0.04$	$\lambda = 0.05$	$\lambda = 0.06$	$\lambda = 0.07$	$\lambda = 0.08$	$\lambda = 0.09$	$\lambda = 0.1$
$T = 0.5$								
$S_0 = 13$	7.2144	7.1783	7.1421	7.1113	7.0753	7.0518	7.0352	7.0239
$S_0 = 15$	5.3459	5.3207	5.2933	5.2728	5.2435	5.2219	5.2026	5.1820
$S_0 = 17$	3.6913	3.6723	3.6551	3.6349	3.6167	3.5976	3.5785	3.5689
$S_0 = 20$	1.8296	1.8219	1.8115	1.8053	1.7917	1.7889	1.7781	1.7696
$S_0 = 22$	1.0484	1.0424	1.0386	1.0302	1.0263	1.0222	1.0158	1.0120
$S_0 = 25$	0.4039	0.4029	0.4008	0.3987	0.3970	0.3955	0.3934	0.3917
$T = 1$								
$S_0 = 13$	7.4999	7.4357	7.3652	7.3024	7.2419	7.1974	7.1660	7.1251
$S_0 = 15$	5.8151	5.7651	5.7085	5.6562	5.6036	5.5553	5.5217	5.4824
$S_0 = 17$	4.3518	4.3081	4.2709	4.2269	4.1898	4.1534	4.1186	4.0907
$S_0 = 20$	2.6553	2.6295	2.6015	2.5767	2.5516	2.5274	2.5086	2.4870
$S_0 = 22$	1.8442	1.8318	1.8123	1.7921	1.7737	1.7611	1.7391	1.7264
$S_0 = 25$	1.0286	1.0186	1.0082	1.0025	0.9903	0.9792	0.9723	0.9628
$T = 2$								
$S_0 = 13$	8.1410	8.0041	7.8523	7.7230	7.6223	7.5280	7.4635	7.4024
$S_0 = 15$	6.6731	6.5589	6.4308	6.3218	6.2252	6.1371	6.0676	5.9986
$S_0 = 17$	5.4104	5.3055	5.2042	5.1168	5.0343	4.9526	4.8748	4.8106
$S_0 = 20$	3.8666	3.7982	3.7256	3.6529	3.5859	3.5336	3.4735	3.4185
$S_0 = 22$	3.0658	3.0102	2.9488	2.8976	2.8444	2.7933	2.7484	2.7049
$S_0 = 25$	2.1439	2.1075	2.0634	2.0265	1.9838	1.9503	1.9171	1.8804
$T = 3$								
$S_0 = 13$	8.7434	8.5121	8.2875	8.1108	7.9558	7.8369	7.7318	7.6422
$S_0 = 15$	7.4200	7.2249	7.0267	6.8641	6.7122	6.5916	6.4825	6.3747
$S_0 = 17$	6.2585	6.0947	5.9232	5.7791	5.6455	5.5291	5.4201	5.3231
$S_0 = 20$	4.8275	4.6914	4.5649	4.4392	4.3335	4.2328	4.1410	4.0585
$S_0 = 22$	4.0485	3.9348	3.8260	3.7209	3.6255	3.5379	3.4556	3.3781
$S_0 = 25$	3.0941	3.0136	2.9293	2.8432	2.7681	2.6962	2.6376	2.5721

guarantee that the total discount rate is positive, $r + \lambda > 0$. All other parameters are the same – $\sigma = 0.03,\ K = 20$, and the initial asset price is among $S_0 \in \{13,\ 15,\ 17,\ 20,\ 22,\ 25\}$. We can see that the option prices decrease when the discount factor increases. Something more – this influence is larger for the longer maturities.

5. Pricing discounted American call options

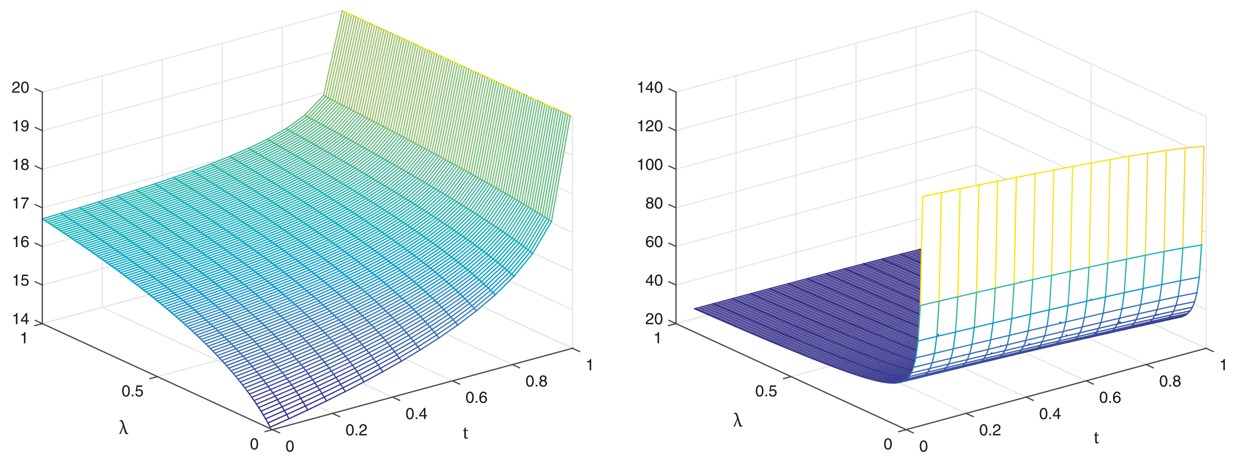
As we can see in Proposition 3.1, the early exercise is never optimal if the discount factor is zero, $\lambda = 0$. Therefore the price of an American call coincides with the price of the corresponding European call. Suppose now that $\lambda > 0$. The algorithm described in Section 4 is appropriate for call options too. We shall present it briefly emphasizing the differences with the put case. Proposition 3.2 gives that the exercise region at time t has the form $[c(t), \infty)$. We use the same intervals as in the put case. The difference is that in the call case the piecewise linear functions for the Brownian motion are above. The corresponding to (4.3) derivative prices which we shall maximize to approximate the exercise boundary now have the form

$$\begin{aligned} &A(x; \{t_0, \dots, t_n\}; \{C_0, \dots, C_n\}) = E^x\left[e^{-(r+\lambda)(\tau\wedge T)}(S_{\tau\wedge T} - K)^+\right] \\ &= E^x\left[e^{-(r+\lambda)\tau}(S_\tau - K)\Lambda_T\right] + e^{-(r+\lambda)T}E^x\left[(S_T - K)^+\Phi_T\right] \\ &= x\sum_{m=1}^{n} E\left[e^{-\alpha_{2,m}\tau} I_{t_{m-1}<\tau\le t_m}\right] - KE\left[e^{-\alpha_1\tau}\Lambda_T\right] \\ &\quad + xe^{-\alpha_3 T}E\left[e^{\sigma B_T} I_{B_T>k, \Phi_T=1}\right] - Ke^{-\alpha_1 T}P(B_T > k, \Phi_T = 1). \end{aligned} \tag{5.1}$$

Hence, the algorithm turns to

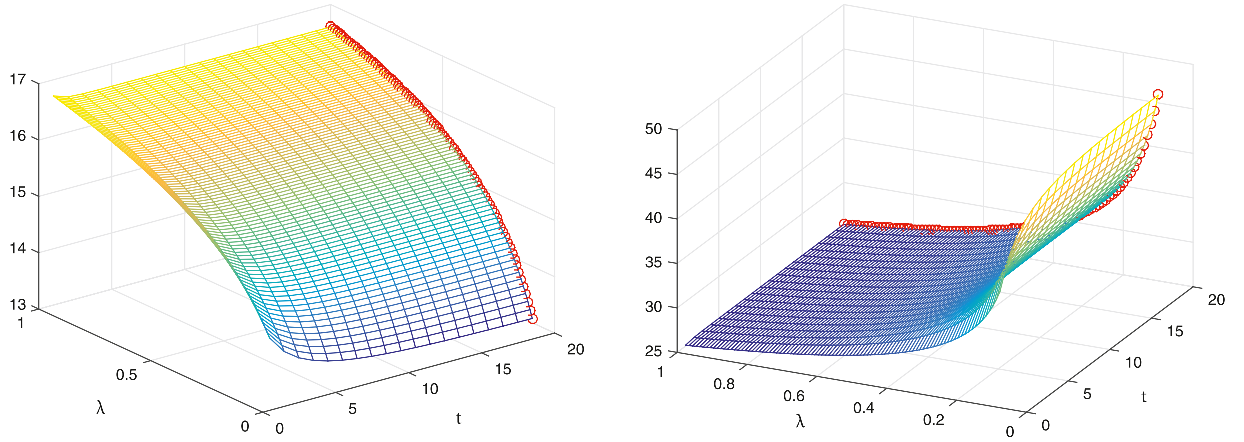
1. The value of the exercise boundary at the maturity is given by Eq. (3.7).
2. For a fixed $x \ge K$, let $C(x)$ be

$$C(x) = \arg\max\{C : A(x; \{0, t_n - t_{n-1}\}; \{C, C_n\})\} \tag{5.2}$$



(a) The put exercise boundary (b) The call exercise boundary

Fig. 2. Early exercise boundary – short time range. The parameters for the put options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0, 1)$, and $T \in (0, 1)$. The values for the call style options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0.01, 1)$, and $T \in (0, 1)$. The algorithm with three steps is used.



(a) The put exercise boundary (b) The call exercise boundary

Fig. 3. Early exercise boundary – long time range. The parameters for the put options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0, 1)$, and $T \in (0, 20)$. The values for the call style options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0.1, 1)$, and $T \in (0, 20)$. The algorithm with three steps is used.

as in Eq. (4.5). Note that to calculate the Laplace transforms in Eq. (5.1) we use Propositions A.3 and A.5 taking in attention that the Brownian motion is a symmetric process. The form of the early exercise region gives us that C_{n-1} can be obtained by each of both formulas

$$\begin{aligned} C_{n-1} &= \min\{x : C(x) = x\} \\ C_{n-1} &= \min\{x : A(x; \{0, t_n - t_{n-1}\}; \{C(x), C_n\}) = x - K\}. \end{aligned} \tag{5.3}$$

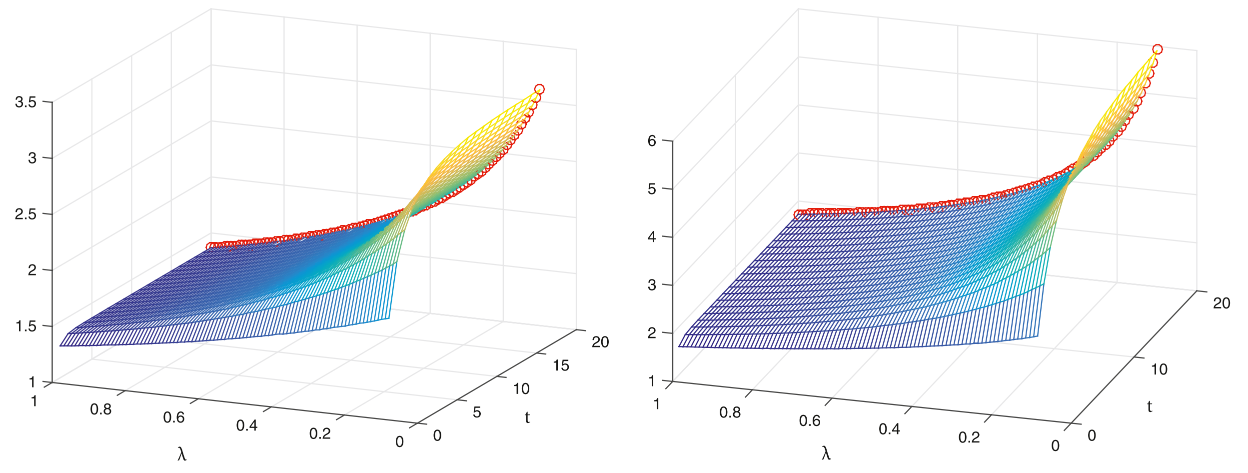
3. Suppose that we have found the values of $C_m, C_{m+1}, \ldots, C_n$ for some $m < n$. For a fixed x we define $C(x)$ by

$$C(x) = \arg\max\{C : A(x; \{0, t_m - t_{m-1}\ldots, t_n - t_{m-1}\}; \{C, C_m, \ldots, C_n\})\}. \tag{5.4}$$

The value of C_{m-1} is given by any of both formulas

$$\begin{aligned} C_{n-1} &= \min\{x : C(x) = x\} \\ C_{n-1} &= \min\{x : A(x; \{0, t_m - t_{m-1}\ldots, t_n - t_{m-1}\}; \{C(x), C_m, \ldots, C_n\}) = x - K\}. \end{aligned} \tag{5.5}$$

In Fig. 2b we present the early exercise boundary varying the discount factor λ between 0.01 and 1 and the time between zero and one. All other parameters have the same values as in Section 4.3 – $r = 0.05$, $\sigma = 0.3$, and $K = 20$. We can see that for small λ's, the exercise boundary has significantly large values. This is in accordance with the fact that the early exercise



(a) The put option prices (b) The call option prices

Fig. 4. American option prices. The used parameters for the put options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0, 1)$, and $T \in (0, 20)$. For the call style options we use values $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0.1, 1)$, and $T \in (0, 20)$. The second Monte Carlo method is used for option pricing.

is never optimal when the discount factor is zero. In Fig. 3b we present the behavior of the early exercise boundary for longer maturities. The discount factor is taken in interval $[0.1, 1]$ whereas the time is between 1 and 20.

Both Monte Carlo methods, which we presented above, can be applied for call options too. The derived prices are presented at Fig. 4b. We use the second Monte Carlo method, since we have seen that it works faster and produces more accurate results. The values of the perpetual calls, obtained by the use of the explicit formulas (6.1) and (6.4) from Section 6, are plotted by red points. It can be seen again that for a large enough t the surfaces tend to the corresponding perpetual values.

In addition to these Monte Carlo methods we present a finite difference approach for solving the Black–Scholes style Eq. (4.10), which in the call case turns to

$$\begin{aligned} &A_t(t,x) + rxA_x(t,x) + \tfrac{1}{2}\sigma^2x^2A_{xx}(t,x) - rA(t,x) = 0, \ \ (t,x) \in (0,T) \times (0,c(t)) \\ &A(t,c(t)) = \exp(-\lambda t)(c(t) - K), \ \ t \in [0,T] \\ &A(T,x) = \exp(-\lambda T)(x-K)^+, \ \ x < c(T). \end{aligned} \tag{5.6}$$

We solve Eq. (5.6) backwards in the region $[0,T] \times [0, c(0)]$. We present below our finite difference algorithm.

1. We divide the region $[0,T] \times [0,c(0)]$ into $M \times N$ nodes. We shall denote by x_n and t_m the values of x and t in the nodes, and by $F(m,n)$ our approximation for the equation solution. Note that $F(1,\cdot)$ are the solutions at the maturity since we work backwards. The length of the divisions will be denoted by Δt and Δx.
2. We calculate $q < M$ uniformly taken points from the optimal exercise boundary.
3. We use a cubic spline interpolation to derive the exercise boundary values at all M points. We shall denote them by c_m, $m = 1, 2, \dots, M$. A similar suggestion can be found in Lee [32].
4. For every $m = 1, 2, \dots, M$ we found the lowest node (m,n) which is above the exercise boundary c_m. We shall denote it by L_m.
5. Lower boundary condition: for every $m = 1, 2, \dots, M$ we state $F(m, 1) = 0$.
6. Upper boundary condition: for every $m = 1, 2, \dots, M$ and $n = L_m, L_{m+1}, \dots, N$ we state $F(m,n) = e^{-\lambda t_m}(x_n - K)$.
7. Right boundary condition: for every $n = 1, 2, \dots, N$ we state $F(1,n) = e^{-\lambda T}(x_n - K)^+$.
8. Suppose that for some m we have derived the solution values $F(j,n)$ for every $n = 1, 2, \dots, N$ and $j = 1, 2, \dots, m-1$. Let $1 < n < L_m$. We discretize Eq. (5.6) in the point (m,n), as

$$\begin{aligned} 0 = &\frac{F(m-1,n) - F(m,n)}{\Delta t} + rx_n\frac{F(m-1,n) - F(m-1,n-1)}{\Delta x} \\ &+\frac{1}{2}\sigma^2x_n^2\frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{(\Delta x)^2} \\ &-rF(m,n). \end{aligned} \tag{5.7}$$

Table 3
American call option prices – high frequency grid. The used parameters are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in \{0.01, 0.04, 0.07, 0.1\}$, and $T \in \{1, 2, 3\}$. The grid for the finite difference approach has $M = 500,000$ time nodes, and $N = 1000$ x-nodes. The first values are obtained by the binomial tree method, the second values by the second Monte Carlo method, and the third values by the finite difference approach.

	$T = 1$	$T = 2$	$T = 3$
$\lambda = 0.01$			
$S_0 = 25$	6.5497/6.5492/6.5467	7.8889 /7.8869/7.8839	8.9776/8.9760/8.9713
$S_0 = 30$	11.0641/11.0634/11.0626	12.2126/12.2112/12.2093	13.2259/13.2247/13.2213
$S_0 = 35$	15.8756/15.8770/15.8750	16.8346/16.8355/16.8326	17.7389/17.7385/17.7357
$S_0 = 40$	20.7842/20.7808/20.7840	21.6013/21.5991/21.6001	22.3989/22.4012/22.3966
$S_0 = 45$	25.7221/25.7212/25.7221	26.4373/26.4371/26.4366	27.1413/27.1443/27.1398
$S_0 = 50$	30.6688/30.6716/30.6688	31.3068/31.3110/31.3064	31.9314/31.9350/31.9303
$\lambda = 0.04$			
$S_0 = 25$	6.3571/6.3541/6.3560	7.4446/7.4439/7.4427	8.2591/8.2628/8.2567
$S_0 = 30$	10.7429/10.7413/10.7424	11.5481/11.5468/11.5469	12.2181/12.2233/12.2164
$S_0 = 35$	15.4283/15.4284/15.4281	15.9655/15.9576/15.9648	16.4717/16.4748/16.4705
$S_0 = 40$	20.2301/20.2361/20.2300	20.5649/20.5630/20.5644	20.9234/20.9244/20.9225
$S_0 = 45$	25.0958/25.0919/25.0958	25.2866/25.2880/25.2863	25.5227/25.5156/25.5221
$S_0 = 50$	30.0181/30.0179/30.0181	30.1060/30.1047/30.1058	30.2437/30.2444/30.2432
$\lambda = 0.07$			
$S_0 = 25$	6.1850/6.1851/6.1841	7.0945/7.0920/7.0931	7.7374/7.7360/7.7356
$S_0 = 30$	10.4893/10.4889/10.4889	11.0874/11.0823/11.0864	11.5667/11.5712/11.5654
$S_0 = 35$	15.1481/15.1496/15.1479	15.4653/15.4639/15.4647	15.7714/15.7697/15.7705
$S_0 = 40$	20.0089/20.0080/20.0088	20.1186/20.1243/20.1183	20.2732/20.2680/20.2726
$S_0 = 45$	25.0000/25.0000/25.0000	25.0003/25.0001/25.0003	25.0316/25.0347/25.0314
$S_0 = 50$	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000
$\lambda = 0.1$			
$S_0 = 25$	6.0440/6.0423/6.0432	6.8218/6.8198/6.8206	7.3406/7.3413/7.3391
$S_0 = 30$	10.3176/10.3187/10.3172	10.7693/10.7720/10.7685	11.1170/11.1151/11.1160
$S_0 = 35$	15.0274/15.0278/15.0273	15.1914/15.1923/15.1910	15.3658/15.3652/15.3651
$S_0 = 40$	20.0000/20.0000/ 20.0000	20.0002/20.0001/20.0002	20.0271/20.0279/20.0269
$S_0 = 45$	25.0000/25.0000/ 25.0000	25.0000/25.0000/25.0000	25.0000/25.0000/25.0000
$S_0 = 50$	30.0000/30.0000/ 30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000

The derivatives F_t, F_x, and F_{xx} are approximated by

$$\begin{aligned} F_t &= \frac{F(m-1,n) - F(m,n)}{\Delta t} \\ F_x &= \frac{F(m-1,n) - F(m-1,n-1)}{\Delta x} \\ F_{xx} &= \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{(\Delta x)^2}, \end{aligned} \tag{5.8}$$

respectively. We derive the value $F(m,n)$ rearranging Eq. (5.7)

$$F(m,n) = \frac{F(m-1,n)}{1+r\Delta t} + \frac{\Delta t}{1+r\Delta t}\left[r x_n \frac{F(m-1,n) - F(m-1,n-1)}{\Delta x} + \frac{1}{2}\sigma^2 x_n^2 \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{(\Delta x)^2}\right]. \tag{5.9}$$

We apply the algorithm above to price call options with the following parameters – the risk free rate is $r = 0.05$, the volatility is $\sigma = 0.3$, the strike is $K = \$20$. We vary the maturity among $T \in \{1,\ 2,\ 3\}$, the discount factor among $\lambda \in \{0.01,\ 0.04,\ 0.07,\ 0.1\}$, and the initial asset value among $S_0 \in \{\$25,\ \$30,\ \$35,\ \$40,\ \$45,\ \$50\}$. The presented for put options results show that $q = 16$ points are quite enough to approximate the exercise boundary. We find the values of the exercise boundary using the four steps algorithm.[4] We divide the time interval into $M = 500,000$ whereas we use a $N = 1000$-division of the x-interval. We compare the derived results with the results obtained by the second Monte Carlo method, which is iterated 100 times and the received results are averaged. Also, both results are compared with the results calculated by the use of the Cox-Ross-Rubinstein binomial tree method with 10,000 steps. The results are presented in Table 3. The first values are obtained using the binomial tree method. The second reported values are calculated through the second Monte Carlo method, whereas the third values are derived by the use of the finite difference approach, presented

[4] We made the same observation as in the put case – the three steps algorithm produces practically the same prices, but the computational time is significantly lower.

Table 4
American call option prices – low frequency grid. The used parameters are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in \{0.01, 0.04, 0.07, 0.1\}$, and $T \in \{1, 2, 3\}$. The grid for the finite difference approach has $M = 20,000$ time nodes, and $N = 250$ x-nodes. First are reported binomial tree values, after them are the values derived by the second Monte Carlo method, and the third values are obtained by the finite difference approach.

	$T = 1$	$T = 2$	$T = 3$
$\lambda = 0.01$			
$S_0 = 25$	6.5497/6.5374/6.5365	7.8889 /7.9319/7.8685	8.9776/8.9786/8.9522
$S_0 = 30$	11.0641/11.0810/11.0580	12.2126/12.2822/12.1993	13.2259/13.2309/13.2075
$S_0 = 35$	15.8756/15.8597/15.8733	16.8346/16.8704/16.8267	17.7389/17.6860/17.7261
$S_0 = 40$	20.7842/20.7670/20.7833	21.6013/21.5840/21.5967	22.3989/22.4813/22.3901
$S_0 = 45$	25.7221/25.7044/25.7218	26.4373/26.4136/26.4346	27.1413/27.0754/27.1354
$S_0 = 50$	30.6688/30.6498/30.6686	31.3068/31.3738/31.3052	31.9314/31.9680/31.9273
$\lambda = 0.04$			
$S_0 = 25$	6.3571/6.3715/6.3525	7.4446/7.4297/7.4372	8.2591/8.3046/8.2496
$S_0 = 30$	10.7429/10.8055/10.7407	11.5481/11.5695/11.5432	12.2181/12.1778/12.2109
$S_0 = 35$	15.4283/15.4322/15.4273	15.9655/15.9233/15.9623	16.4717/16.4139/16.4663
$S_0 = 40$	20.2301/20.2321/20.2296	20.5649/20.5862/20.5628	20.9234/20.9299/20.9194
$S_0 = 45$	25.0958/25.1078/25.0955	25.2866/25.2922/25.2852	25.5227/25.5169/25.5198
$S_0 = 50$	30.0181/30.0024/30.0179	30.1060/30.1112/30.1051	30.2437/30.2085/30.2416
$\lambda = 0.07$			
$S_0 = 25$	6.1850/6.1967/6.1814	7.0945/7.1031/7.0888	7.7374/7.7462/7.7300
$S_0 = 30$	10.4893/10.5242/10.4875	11.0874/11.0556/11.0834	11.5667/11.5803/11.5610
$S_0 = 35$	15.1481/15.1647/15.1472	15.4653/15.4483/15.4627	15.7714/15.7806/15.7673
$S_0 = 40$	20.0089/20.0005/20.0086	20.1186/20.1092/20.1172	20.2732/20.2294/20.2705
$S_0 = 45$	25.0000/25.0000/25.0000	25.0003/25.0012/25.0001	25.0316/25.0000/25.0304
$S_0 = 50$	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000
$\lambda = 0.1$			
$S_0 = 25$	6.0440/6.0455/6.0409	6.8218/6.8255/6.8168	7.3406/7.3565/7.3344
$S_0 = 30$	10.3176/10.2889/10.3160	10.7693/10.7896/10.7658	11.1170/11.1333/11.1123
$S_0 = 35$	15.0274/15.0205/15.0269	15.1914/15.2057/15.1895	15.3658/15.3476/15.3627
$S_0 = 40$	20.0000/20.0000/ 20.0000	20.0002/20.0000/20.0000	20.0271/20.0267/20.0259
$S_0 = 45$	25.0000/25.0000/ 25.0000	25.0000/25.0000/25.0000	25.0000/25.0000/25.0000
$S_0 = 50$	30.0000/30.0000/ 30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000

Table 5
Computational times in seconds.

	Time
Deriving a point from the exercise boundary – three points algorithm	0.8698
Deriving a point from the exercise boundary – four points algorithm	23.5644
Deriving a point from the exercise boundary – five points algorithm	437.8964
Pricing – first Monte Carlo method, 100 iterations	182.0521
Pricing – first Monte Carlo method, 1 iterations	1.5644
Pricing – second Monte Carlo method, 100 iterations	143.0678
Pricing – second Monte Carlo method, 1 iterations	1.3534
Pricing – finite difference method, high frequency	104.4804
Pricing – finite difference method, low frequency	0.9852

above. We can see that the prices which produces the Monte Carlo method is very near to the real ones, but the finite difference approach gives values which are almost identical to the binomial tree prices. We use a very frequent discretization above to obtain as accurate results as possible. This leads to some increase in the computational time. In practice, it is enough to use values $M = 20,000$ and $N = 250$ to obtain sufficiently accurate results – we report them in Table 4. Also, we present in this table the Monte Carlo results with only one iteration. We can observe that the finite difference approach derives very accurate results, whereas the Monte Carlo method exhibits some fluctuations.

The computational times for all algorithms are presented in Table 5. The necessary time for deriving exercise boundaries and for the Monte Carlo method is the same as for the put style options. We can see that the finite difference method works very fast and produces extremely accurate prices. Something more – the finite difference approach allows deriving the whole price structure w.r.t. the time as well as w.r.t. the initial asset price by one iteration.

6. Perpetual American options

Now we shall examine options without maturity restrictions. This means that $T = \infty$ and the set of the moments in which exercising is possible is $\mathcal{U} \equiv \mathbb{R}^+$. As we already mentioned, in the perpetual case the early exercise boundary is flat – so we can suppose that it is a constant denoted by c.

We shall first consider the call case. Suppose that $t = 0$. We shall denote by $A(x)$ the price of the option if the underlying asset has initial value x. If the discount rate is zero, Proposition 3.1 and remark 3.1 lead to the conclusion that the early exercise is never optimal. Therefore the price of a perpetual American call can be derived letting T to tend to infinity in the Black–Scholes formula for the European option. Therefore $A(x) = x$. Suppose now that $\lambda > 0$. The following theorem holds.

Theorem 6.1. *The price of a discounted American perpetual call option written on the asset with initial price below the exercise boundary $S_0 = x < c$ and strike price K is*

$$A(x) = \left(\frac{x}{\gamma}\right)^{\gamma}\left(\frac{\gamma-1}{K}\right)^{\gamma-1}, \tag{6.1}$$

where

$$\gamma = \sqrt{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2}-\frac{1}{2}\right). \tag{6.2}$$

If the initial asset value is above the exercise boundary, then the option price is

$$A(x) = x - K. \tag{6.3}$$

The exercise boundary value is

$$c = \frac{\gamma}{\gamma-1}K \tag{6.4}$$

and the optimal stopping time is the first hitting moment to the interval $[c, \infty)$.

Proof. Since the asset price can be written as

$$S_t = e^{\ln x + \left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t}, \tag{6.5}$$

the stopping time can be viewed as the first hitting time of the Brownian motion with drift

$$\mu = \frac{r}{\sigma} - \frac{\sigma}{2} \tag{6.6}$$

to the value

$$a = \frac{\ln c - \ln x}{\sigma} > 0. \tag{6.7}$$

Using Eq. (A.8) from Proposition A.6 we derive

$$\begin{aligned}
&A(x;c) = E^x\left[e^{-(r+\lambda)\tau}(S_\tau - K)I_{\tau<\infty}\right] \\
&= (c-K)E^x\left[e^{-(r+\lambda)\tau}I_{\tau<\infty}\right] \\
&= (c-K)\exp\left\{-\left(\sqrt{\left(\tfrac{r}{\sigma}-\tfrac{\sigma}{2}\right)^2 + 2(r+\lambda)} - \left(\tfrac{r}{\sigma}-\tfrac{\sigma}{2}\right)\right)\tfrac{\ln c - \ln x}{\sigma}\right\} \\
&= (c-K)\exp\left\{-\left(\sqrt{\left(\tfrac{r}{\sigma^2}-\tfrac{1}{2}\right)^2 + 2\tfrac{r+\lambda}{\sigma^2}} - \left(\tfrac{r}{\sigma^2}-\tfrac{1}{2}\right)\right)(\ln c - \ln x)\right\} \\
&= (c-K)\left(\tfrac{c}{x}\right)^{-\gamma}.
\end{aligned} \tag{6.8}$$

Elementary calculations show that function (6.8) has maximum w.r.t. the variable c when its value is just (6.4). Therefore, equation (6.8) leads that the option price is

$$\begin{aligned}
c(x) &= (c-K)\left(\tfrac{c}{x}\right)^{-\gamma} \\
&= \left(\tfrac{x}{\gamma}\right)^{\gamma}\left(\tfrac{\gamma-1}{K}\right)^{\gamma-1}
\end{aligned}$$

as in Eq. (6.1). □

Remark 6.1. Let us see what happens when $\lambda = 0$. Eq. (6.2) leads to $\gamma = 1$ and therefore equation (6.4) turns to $c = \infty$. So, the optimal stopping time does not exist as we already proved in Proposition 3.1. Also, formula (6.1) turns to $A(x) = x$.

Analogously, we can prove a theorem for the price of a discounted perpetual American put, which we denote by $B(x)$.

Theorem 6.2. *If the initial asset price is above the early exercise boundary, $S_0 = x > c$, then the price of a discounted American perpetual put is*

$$B(x) = \left(\frac{K}{\gamma+1}\right)^{\gamma+1}\left(\frac{\gamma}{x}\right)^{\gamma}, \tag{6.9}$$

where

$$\gamma = \sqrt{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2}-\frac{1}{2}\right). \tag{6.10}$$

If the initial asset value is below the exercise boundary, then the option price is

$$B(x) = K - x. \tag{6.11}$$

The exercise boundary is

$$c = \frac{\gamma}{\gamma+1}K \tag{6.12}$$

and the optimal stopping time is the moment of first hitting to the interval $[0, c]$.

Proof. Analogously to the proof of Theorem 6.1, we see that the stopping time is the first hitting time of the Brownian motion with drift (6.6) to the value (6.7), which in this case is negative. So we have to use Eq. (A.9) to derive

$$\begin{aligned} B(x;c) &= E^x\left[e^{-(r+\lambda)\tau}(K-S_\tau)I_{\tau<\infty}\right] \\ &= (K-c)E^x\left[e^{-(r+\lambda)\tau}I_{\tau<\infty}\right] \\ &= (K-c)\exp\left\{\left(\sqrt{\left(\tfrac{r}{\sigma}-\tfrac{\sigma}{2}\right)^2+2(r+\lambda)}+\left(\tfrac{r}{\sigma}-\tfrac{\sigma}{2}\right)\right)\frac{\ln c-\ln x}{\sigma}\right\} \\ &= (K-c)\left(\tfrac{c}{x}\right)^{\gamma}. \end{aligned} \tag{6.13}$$

Formula (6.13) has maximum when c is as in Eq. (6.12). Therefore Eq. (6.13) turns to Eq. (6.9). □

The following proposition is a corollary from Theorems 6.1 and 6.2 and gives the option prices at some moment $t > 0$.

Corollary 6.1. *The prices of live perpetual American call and put options in moment $t > 0$ are given by $A(t,x) = e^{-\lambda t}A(x)$ and $B(t,x) = e^{-\lambda t}B(x)$, respectively. The functions $A(x)$ and $B(x)$ are given by Eqs. (6.1) and (6.9).*

Proof. We shall give the proof for a call option. Since the asset price is driven by a Markov process, we have for the price of a live option at time $t > 0$

$$\begin{aligned} A(t,x) &= E^{t,x}\left[e^{-r(\tau-t)}e^{-\lambda\tau}(S_\tau-K)^+\right] \\ &= e^{-\lambda t}E^{t,x}\left[e^{-(\lambda+r)(\tau-t)}(c-K)\right] \\ &= e^{-\lambda t}A(x). \end{aligned} \tag{6.14}$$

We can prove the proposition for the put option in the same way. □

Remark 6.2. We can easily check that the functions $A(t,x)$ and $B(t,x)$ are the solutions of the boundary value problems

$$\begin{aligned} & f_t(t,x) + \mathcal{A}f(t,x) - rf(t,x) = 0 \\ & f(t,c) = e^{-\lambda t}\frac{K}{\gamma \mp 1} \equiv e^{-\lambda t}\begin{cases} c-K \\ or \\ K-c. \end{cases} \end{aligned} \tag{6.15}$$

The equations for the call and put options hold in the regions $(0,c)$ and (c,∞), respectively. Also, we can easily check that the functions $A(t,x)$ and $B(t,x)$ are continuously differentiable in the point c. This is not accidental – it is a manifestation of the so-called smooth fit principle, which gives the relation between optimal stopping and variational inequalities – see Brekke and Øksendal [8], McKean [39], Shiryaev [43], or Bensoussan and Lions [5].

7. Conclusions

In this article we have examined the problem of pricing discounted American options. We first approximate the early exercise boundary by exponent of piecewise linear functions. This turns the free boundary problem for American option pricing to a boundary value problem with known boundaries. We presented two Monte Carlo methods for deriving the option prices as well as a finite difference approach. The reported results confirm the consistency of our approach. We also conclude that the second Monte Carlo method works and converges faster than the first one, but the finite difference method produces very accurate results for a very small computational time. Also, we can conclude that relatively small numbers of the grid nodes and the algorithm steps produce very good results. This diminishes significantly the computation time. We also derived closed form formulas for perpetual discounted American options.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Tsvetelin S. Zaevski: Conceptualization, Methodology, Formal analysis, Software, Investigation, Writing - original draft, Writing - review & editing.

Acknowledgments

The author would like to express his special thanks of gratitude to professor Racho Denchev for the many advices which he gave me during the work on this article.

The present research has been supported by the project KP-06-N32/8 with Bulgarian National Science Fund.

The author would like to express sincere gratitude to the Editors and the anonymous reviewers for the helpful and constructive comments which substantially improve the quality of this paper.

Appendix A. First hitting time propositions

We shall present some propositions related to the first hitting time of a Brownian motion to a continuous piecewise linear function $c(t) = (a_m t + b_m) I_{t_{m-1}<t\le t_m}$ w.r.t. the time grid $0 \equiv t_0 < t_1 < \ldots < t_n \equiv T$. Let the values at the nodes be $c_m = a_m t_m + b_m$. Due to the symmetry of the Brownian motion, we may assume that $c(t)$ starts from a negative value, $c(0) < 0$. If $c(0) > 0$, then we have to examine the opposite function $\bar{c}(t) = -c(t)$. Note that the most of the presented propositions are proven in Zaevski [53] for the upper hitting problem, $c(0) > 0$. If the function is linear we shall use the notation $c(t) = at + b,\ b < 0$. We denote by $N(\cdot)$ the cumulative distribution function of the standard normal distribution.

Proposition A.1. *The probability that the Brownian motion hits the linear function before the moment T is given by the equation*

$$g(T; a, b) \equiv P(\tau < T) = N\left(\frac{aT + b}{\sqrt{T}}\right) + \exp(-2ab) N\left(\frac{-aT + b}{\sqrt{T}}\right). \tag{A.1}$$

Proof. The proof can be found in Wang and Pötzelberger [44], equation (3), Karatzas and Shreve [25], or in Zaevski [53], Proposition 3.1. □

Proposition A.2. *Let $\alpha > 0$. Then the truncated Laplace transform of the first hitting time before T is given by the equation*

$$L(T, \alpha; a, b) = E\left[e^{-\alpha\tau} \Lambda_T\right] = e^{-b\left(\sqrt{a^2+2\alpha}+a\right)} g\left(T; -\sqrt{a^2 + 2\alpha}, b\right), \tag{A.2}$$

where $g(\cdot)$ is the function from Eq. (A.1).

Proof. The proof can be found in Zaevski [53], theorem 3.1. □

Proposition A.3. *Let $\alpha > 0$. Then the truncated Laplace transform in the interval (t_{m-1}, t_m) of the first hitting time to the piece-wise linear function is given by the equation*

$$E\left[e^{-\alpha\tau} I_{\tau\in(t_{m-1},t_m)}\right] = \int\limits_{c_1,\ldots,c_{m-1}}^{\infty} \left(\begin{array}{c} \prod\limits_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod\limits_{i=1}^{m-1} \frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{-\alpha t_{m-1}} L(t_m - t_{m-1}, \alpha; a_m, c_{m-1} - x_{m-1}) \end{array}\right) dx_1 \ldots dx_{m-1}, \tag{A.3}$$

where $L(\cdot)$ is the function from Eq. (A.2).

Proof. The proof can be found in Zaevski [53], theorem 4.1. □

Proposition A.4. *Let $z > c(T)$ and the function be linear. Then we have*

$$V(\alpha, z, T; c(\cdot)) \equiv E\left[e^{\alpha B_T} I_{B_T<z,\Phi_T=1}\right] = \exp\left(\frac{T\alpha^2}{2}\right)\left[N\left(\frac{z-T\alpha}{\sqrt{T}}\right) - N\left(\frac{c(T)-T\alpha}{\sqrt{T}}\right) - e^{2b(\alpha-a)}\left(N\left(\frac{z-T\alpha-2b}{\sqrt{T}}\right) - N\left(\frac{c(T)-T\alpha-2b}{\sqrt{T}}\right)\right)\right].$$

(A.4)

Proof. The proof can be found in Zaevski [53], theorem 3.2. □

Corollary A.1. *We have for $z > c(T)$*

$$U(z,T;c(\cdot))\equiv P(B_T < z, \Phi_T = 1) = N\left(\frac{z}{\sqrt{T}}\right) - N\left(\frac{c(T)}{\sqrt{T}}\right) - e^{-2ba}\left[N\left(\frac{z-2b}{\sqrt{T}}\right) - N\left(\frac{c(T)-2b}{\sqrt{T}}\right)\right]. \tag{A.5}$$

Proof. We have to rewrite formula (A.4) for $\alpha = 0$. □

Proposition A.5. *If the boundary function is piecewise linear, then the corresponding to (A.4) and (A.5) formulas are*

$$E\left[e^{\alpha B_T} I_{B_T<z,\Phi_T=1}\right] = \int_{c_1,\dots,c_{n-1}}^{\infty} \left(\begin{array}{c} \prod_{i=1}^{n-1}\left(1-\exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\alpha x_{n-1}} V(\alpha, z-x_{n-1}, t_n-t_{n-1}; c_{n-1}(\cdot)-x_{n-1}) \end{array} \right) dx_1\dots dx_{n-1}, \tag{A.6}$$

and

$$P(B_T < z, \Phi_T = 1) = \int_{c_1,\dots,c_{n-1}}^{\infty} \left(\begin{array}{c} \prod_{i=1}^{n-1}\left(1-\exp\left(-\frac{2(c_{i-1}-x_{i-1})(c_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\alpha x_{n-1}} U(z-x_{n-1}, t_n-t_{n-1}; c(\cdot)-x_{n-1}) \end{array} \right) dx_1\dots dx_{n-1} \tag{A.7}$$

where the functions $V(\cdot)$ and $U(\cdot)$ are given by Eqs. (A.4) and (A.5).

Proof. The proof of the first statement can be found in Zaevski [53], theorem 4.2. The second statement is obtained when $\alpha = 0$. □

The following formulas can be find in Borodin and Salminen [7], page 223, (2.0.1).

Proposition A.6. *Let τ be the first hitting time of a Brownian motion with drift μ to the positive level a. Then the Laplace transform of its distribution is*

$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a}. \tag{A.8}$$

If the value a is negative, then the Laplace transform is

$$E\left[e^{-y\tau} I_{\tau<\infty}\right] = e^{\left(\sqrt{\mu^2+2y}+\mu\right)a}. \tag{A.9}$$

References

[1] Alghalith M. Pricing the American options using the Black–Scholes pricing formula. Phys A 2018;507:443–5.
[2] Armerin F. American perpetual options with random start. Results Appl Math 2019;3:100017. doi:10.1016/j.rinam.2019.100017.
[3] Balajewicz M, Toivanen J. Reduced order models for pricing European and American options under stochastic volatility and jump-diffusion models. J Comput Sci 2017;20:198–204.
[4] Barone-Adesi G, Whaley R. Efficient analytic approximation of American option values. J Finance 1987;42(2):301–20.
[5] Bensoussan A, Lions J. Applications of variational inequalities in stochastic control. Studies in Mathematics and its Applications. Elsevier Science; 2011. ISBN 9780080875330. https://books.google.bg/books?id=B5W6uSrIgH0C
[6] Bjerksund P, Stensland G. Closed-form approximation of American options. Scand J Manag 1993;9:S87–99.
[7] Borodin A, Salminen P. Handbook of Brownian motion - facts and formulae. Probability and Its Applications. Birkhäuser Basel; 2015. ISBN 9783764367053. https://books.google.bg/books?id=8cZrllyXfOUC
[8] Brekke K., Øksendal B.. The high contact principle as a sufficiency condition for optimal stopping. Preprint series: Pure mathematics http://urn nb no/URN: NBN: no-80761990.
[9] Carr P, Jarrow R, Myneni R. Alternative characterizations of American put options. Math Finance 1992;2(2):87–106. doi:10.1111/j.1467-9965.1992.tb00040.x.
[10] Chan T. Hedging and pricing early-exercise options with complex Fourier series expansion. North Am J Econ Finance 2019. doi:10.1016/j.najef.2019.04.016.
[11] Chan T. An SFP–FCC method for pricing and hedging early-exercise options under Lévy processes. Quant Finance 2020;20(8):1325–43. doi:10.1080/14697688.2020.1736322.
[12] Chen W, Du K, Qiu X. Analytic properties of American option prices under a modified Black–Scholes equation with spatial fractional derivatives. Phys A 2018;491:37–44.
[13] Company R, Egorova VN, Jódar L. A front-fixing ETD numerical method for solving jump–diffusion American option pricing problems. Math Comput Simul 2020. doi:10.1016/j.matcom.2020.07.015.

[14] Cox J, Ross S, Rubinstein M. Option pricing: a simplified approach. J Financ Econ 1979;7(3):229–63.
[15] Gao Y, Song H, Wang X, Zhang K. Primal-dual active set method for pricing American better-of option on two assets. Commun Nonlinear Sci Numer Simul 2020;80:104976. doi:10.1016/j.cnsns.2019.104976.
[16] Geske R, Johnson H. The American put option valued analytically. J Finance 1984;39(5):1511–24.
[17] Gong X, Zhuang X. American option valuation under time changed tempered stable Lévy processes. Phys A 2017;466:57–68.
[18] Ho T, Stapleton R, Subrahmanyam M. A simple technique for the valuation and hedging of American options. J Deriv 1994;2(1):52–66.
[19] Huang J, Subrahmanyam M, Yu G. Pricing and hedging American options: a recursive integration method. Rev Financ Stud 1996;9(1):277–300.
[20] Jacka SD. Optimal stopping and the American put. Math Finance 1991;1(2):1–14. doi:10.1111/j.1467-9965.1991.tb00007.x.
[21] Jeon J, Huh J, Park K. An analytic approximation for valuation of the American option under the Heston model in two regimes. Comput Econ 2020;56:499–528. doi:10.1007/s10614-019-09939-2.
[22] Johnson H. An analytic approximation for the American put price. J Financ Quant Anal 1983;18(1):141–8.
[23] Ju N. Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. Rev Financ Stud 1998;11(3):627–46.
[24] Kang M, Jeon J, Han H, Lee S. Analytic solution for American strangle options using Laplace–Carson transforms. Commun Nonlinear Sci Numer Simul 2017;47:292–307.
[25] Karatzas I, Shreve S. Brownian motion and stochastic calculus. Graduate Texts in Mathematics. Springer New York; 1991. ISBN 9780387976556. https://books.google.bg/books?id=ATNy_Zg3PSsC
[26] Kim I. The analytic valuation of American options. Rev Financ Stud 1990;3(4):547–72. http://www.jstor.org/stable/2962115
[27] Klimsiak T, Rozkosz A, Ziemkiewicz B. Valuing American options by simulation: a BSDEs approach. Math Comput Simul 2016;123:1–18. doi:10.1016/j.matcom.2015.11.009.
[28] Kozpınar S, Uzunca M, Karasözen B. Pricing European and American options under Heston model using discontinuous Galerkin finite elements. Math Comput Simul 2020;177:568–87. doi:10.1016/j.matcom.2020.05.022.
[29] Kwok Y-K. Mathematical models of financial derivatives. Springer-Verlag; 1998.
[30] Lamberton D, Lapeyre B. Introduction to stochastic calculus applied to finance. Chapman & Hall/CRC Financial Mathematics Series. Second Edition. Taylor & Francis; 1996. ISBN 9780412718007. https://books.google.bg/books?id=61zI_o-plkEC
[31] Le N, Dang D. Pricing American-style Parisian down-and-out call options. Appl Math Comput 2017;305:330–47.
[32] Lee J-K. A simple numerical method for pricing American power put options. Chaos Solitons Fractals 2020;139:110254. doi:10.1016/j.chaos.2020.110254.
[33] Li C, Ye Y. Pricing and exercising American options: an asymptotic expansion approach. J Econ Dyn Control 2019;107:103729. doi:10.1016/j.jedc.2019.103729.
[34] Lin S, He X-J. A new integral equation approach for pricing American-style barrier options with rebates. J Comput Appl Math 2021;383:113107. doi:10.1016/j.cam.2020.113107.
[35] Longstaff F, Schwartz E. Valuing American options by simulation: a simple least-squares approach. Rev Financ Stud 2001;14(1):113–47.
[36] Lu X, Putri E. A semi-analytic valuation of American options under a two-state regime-switching economy. Phys A 2020;538:122968. doi:10.1016/j.physa.2019.12296.
[37] Madi S, Bouras M, Haiour M, Stahel A. Pricing of American options, using the Brennan–Schwartz algorithm based on finite elements. Appl Math Comput 2018;339:846–52. doi:10.1016/j.amc.2018.06.028.
[38] Magirou EF, Vassalos P, Barakitis N. A policy iteration algorithm for the American put option and free boundary control problems. J Comput Appl Math 2020;373:112544. doi:10.1016/j.cam.2019.112544. Numerical Analysis and Scientific Computation with Applications
[39] McKean H. A free boundary problem for the heat equation arising from a problem in mathematical economics. Ind Manag Rev 1965;6(2):32–9.
[40] Park K, Jeon J. A simple and fast method for valuing American knock-out options with rebates. Chaos Solitons Fractals 2017;103:364–70.
[41] Pascucci A. Free boundary and optimal stopping problems for American Asian options. Finance Stoch 2008;12(1):21–41. doi:10.1007/s00780-007-0051-7.
[42] Rad J, Parand K, Ballestra L. Pricing European and American options by radial basis point interpolation. Appl Math Comput 2015;251:363–77.
[43] Shiryaev A. Optimal stopping rules. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg; 2009. ISBN 9783540841814. https://books.google.bg/books?id=LWFlawEACAAJ
[44] Wang L, Pötzelberger K. Boundary crossing probability for Brownian motion and general boundaries. J Appl Probab 1997;34(1):54–65.
[45] Wong D. Generalized optimal stopping problems and financial markets. Chapman & Hall/CRC Research Notes in Mathematics Series. Taylor & Francis; 1996. ISBN 9780582304000. https://books.google.bg/books?id=uQdW8tADrsAC
[46] Yang Z. Default probability of american lookback option in a mixed jump-diffusion model. Phys A 2020;540:123242. doi:10.1016/j.physa.2019.123242.
[47] Yoon J. Pricing perpetual American options under multiscale stochastic elasticity of variance. Chaos Solitons Fractals 2015;70:14–26.
[48] Yousuf M, Khaliq A, Alrabeei S. Solving complex PIDE systems for pricing American option under multi-state regime switching jump–diffusion model. Comput Math Appl 2018;75(8):2989–3001. doi:10.1016/j.camwa.2018.01.026.
[49] Yu X, Xie X. Pricing American options: RNMs-constrained entropic least-squares approach. North Am J Econ Finance 2015;31:155–73.
[50] Zaevski T. Early exercise boundary of an American put. Comptes Rendus Acad Bulgare Sci 2019;72(6):720–6.
[51] Zaevski T. A new form of the early exercise premium for American type derivatives. Chaos Solitons Fractals 2019;123:338–40. doi:10.1016/j.chaos.2019.04.024.
[52] Zaevski T. Discounted perpetual game call options. Chaos Solitons Fractals 2020;131:109503. doi:10.1016/j.chaos.2019.109503.
[53] Zaevski T. Laplace transforms for the first hitting time of a Brownian motion. Comptes Rendus Acad Bulgare Sci 2020;73(7):934–41. doi:10.7546/CRABS.2020.07.05.
[54] Zhao H, Yang H. Semismooth Newton methods with domain decomposition for American options. J Comput Appl Math 2018;337:37–50. doi:10.1016/j.cam.2017.12.046.